Ranking the Rows of a Permuted Isotonic Matrix in Noise

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Presented at SODA24 [https://arxiv.org/abs/2310.01133],
Joint work with

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Motivations for Ranking

Crowdsourcing data

Tournaments

Motivations for Ranking

Crowdsourcing data

- Correctness of answer for pairs of expert/questions

Tournaments

Motivations for Ranking

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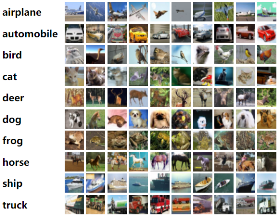
- Correctness of answer for pairs of expert/questions

CIFAR10H Dataset



Tournaments

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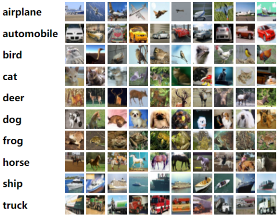

Tournaments

- Pairwise comparison between players

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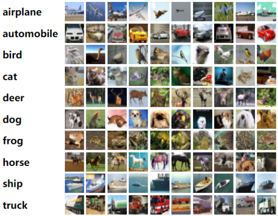

Tournaments

- Pairwise comparison between players

Sports and video games

e.g. [Cattelan et al., 2013]

General Question

Given the correctness of answer from n experts to d questions.

- How accurately can we recover the **ranking of the experts?**

Illustration

Illustration

9 questions

$$
\text{4 experts}\quad \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}
$$

1: Correct answer   0: Wrong answer

Illustration

9 questions

4 experts

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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Good Experts   **Bad Experts**

Illustration

9 questions

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1: Correct answer   0: Wrong answer

Good Experts   **Bad Experts**

Goal: **Ranking of Experts**

1: Correct   0: Wrong

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expert i is correct at question k

$$\Leftrightarrow Y_{ik} = 1$$

Experts/Questions Setting in Crowdsourcing

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Experts $i \in \{1, \ldots, n\}$
Questions $k \in \{1, \ldots, d\}$
We observe for all i, k:

$$Y_{ik} \sim \text{Bern}\left(M_{ik}\right)$$

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- $M_{ik} = 1/2$: random choice of expert i at question k

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$$\Leftrightarrow Y_{ik} = 1$$

We will often assume that $n = d$

Pairwise Comparison Setting in Tournaments

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Player $i \in \{1, \ldots, n\}$
Player $k \in \{1, \ldots, n\}$
We observe for all i, k:

$$Y_{ik} = 1 - Y_{ki} \sim \mathrm{Bern}\left(M_{ik}\right)$$

1: Wins 0: Loses

$$\begin{pmatrix} \times & 1 & 1 \\ 0 & \times & 0 \\ 0 & 1 & \times \end{pmatrix}$$

3 players

Player i wins against player k

$$\Leftrightarrow Y_{ik} = 1$$

Observation Model

$Y_{ik} = \mathrm{Bern}(M_{ik})$, with $M \in [0,1]^{n\times d}$

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- $\mathrm{Poisson}(\lambda)$ observations per entry (i, k)

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- $\lambda \leq 1$: partial observations
- For simplicity, we assume that $\lambda = 1$

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Isotonic Model: [Flammarion et al., 2019]

$\exists$ unknown permutation π^*: l

M_{π^*} is isotonic

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M_{π^*} is isotonic

$$M_{\pi^*} = \begin{pmatrix} 0.9 & 0.8 & 0.9 & 1 \\ 0.8 & 0.7 & 0.9 & 0.8 \\ 0.6 & 0.7 & 0.7 & 0.6 \\ 0.5 & 0.7 & 0.5 & 0.6 \end{pmatrix}$$

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- Goal: recover π^*, the ranking of the experts/rows

Isotonic model

Observation Model

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Shape constraints:

- Increasing Columns **for an unknown permutation** π^*

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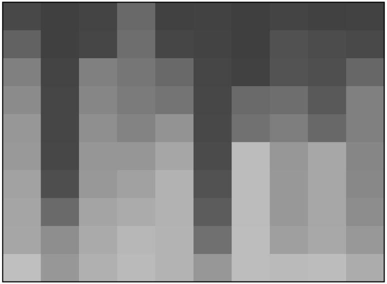

Matrix M_{π^*}. (isotonic)

Isotonic model

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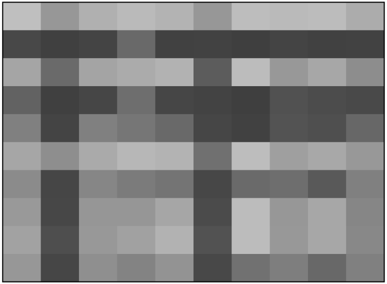

Matrix M (isotonic up to a permutation of rows)

Isotonic model

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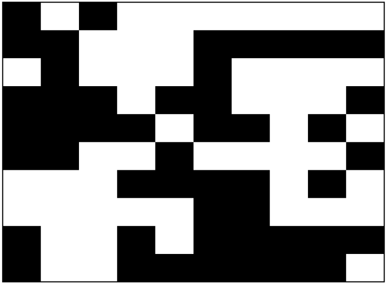

Matrix Y (M in noise)

Isotonic model

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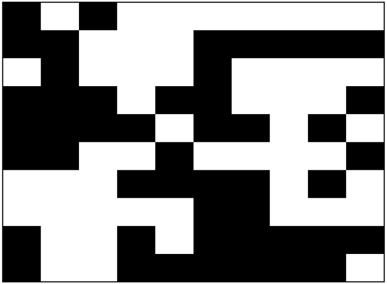
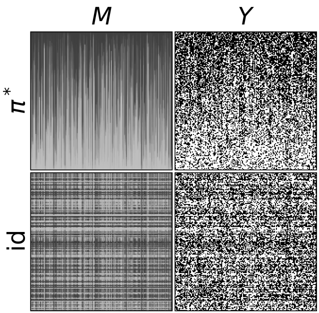
Aim

Estimation of π^*



Matrix Y (M in noise)

Illustration



Other Models

Parametric Models

Non-Parametric Models

Other Models

Parametric Models

- BTL: $M_{ik} = \phi(a_i - b_k)$ ~[Bradley and Terry, 1952]
  - a_i: abilities of the experts
  - b_k: difficulties of the questions

Non-Parametric Models

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Non-Parametric Models

- Isotonic: M_{π^*} is isotonic
- Bi-Isotonic: $M_{\pi^*\eta^*}$ is bi-isotonic

  ~[Shah et al., 2016],[Mao et al., 2018],[Liu and Moitra, 2020]

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  - The rows are increasing up to an unknown permutation η^*

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Observation Model

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Error Measures

Isotonic Model

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Error Measures

Permutation Loss

For an estimator $\hat{\pi}$ of π^*

$$\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2$$

$\|A\|_F^2 = \sum_{i,k} A_{ik}^2$

Isotonic Model

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$$\frac{1}{2} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h & h & h \end{pmatrix}$$

If the two lines are misclassified:
$\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2 = 6h^2$

Isotonic Model

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Reconstruction Loss

For an estimator $\hat{M}$ of M

$$\|\hat{M} - M\|_F^2$$

Isotonic Model

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Shape Constraints:

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Error Measures

Permutation Risk

For an estimator $\hat{\pi}$ of π^*

$$\mathbb{E}\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2$$

Reconstruction Risk

For an estimator $\hat{M}$ of M

$$\mathbb{E}\|\hat{M} - M\|_F^2$$

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Estimation of π^*

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Estimation of π^*

Max Risks

Max Risks

Permuation Estimation:

$$\mathcal{R}^{\hat{\pi}}_{\text{perm}}(n,d) = \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

Max Risks

Permuation Estimation:

$$\mathcal{R}^{\hat{\pi}}_{\text{perm}}(n,d) = \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

- **Supremum** over all isotonic matrices $M_{\pi^*} \in [0,1]^{n\times d}$

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- $\mathcal{R}^{\hat{\pi}}_{\text{perm}}(n,d,\lambda)$ when partial observations

Max Risks

Permuation Estimation:

$$\mathcal{R}^{\hat{\pi}}_{\text{perm}}(n,d) = \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

Matrix Reconstruction:

$$\mathcal{R}^{\hat{M}}_{\text{reco}}(n,d) = \sup_{M,\pi^*} \mathbb{E}[\|\hat{M} - M\|_F^2]$$

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Minimax Risks

Permuation Estimation:

Matrix Reconstruction:

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Minimax Risks

Permuation Estimation:

$$\mathcal{R}^*_{\text{perm}}(n,d) =$$

Matrix Reconstruction:

$$\mathcal{R}^*_{\text{reco}}(n,d) =$$

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Minimax Risks

Permuation Estimation:

$$\mathcal{R}^*_{\text{perm}}(n,d) = \inf_{\hat{\pi}} \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

Matrix Reconstruction:

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- **Supremum** over all isotonic matrices $M_{\pi^*} \in [0,1]^{n\times d}$
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Main Questions

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- Is there a **computational-statistical gap**?

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- Is there a **computational-statistical gap**?
- Is estimating π^* easier than reconstructing M?

Main Questions

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$$\mathcal{R}^*_{\text{perm}}(n,d) = \inf_{\hat{\pi}} \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

Matrix Reconstruction:

$$\mathcal{R}^*_{\text{reco}}(n,d) = \inf_{\hat{M}} \sup_{M,\pi^*} \mathbb{E}[\|\hat{M} - M\|_F^2]$$

- Is there a **computational-statistical gap**?
- Is estimating π^* easier than reconstructing M? ($\mathcal{R}^*_{\text{perm}} \ll \mathcal{R}^*_{\text{reco}}$?)

Short Story

Parametric Models

Non-Parametric Models

Short Story

Parametric Models

- BTL: $M_{ik} = \phi(a_i - b_k)$

Non-Parametric Models

- No computational gap for **parametric models**

  e.g. [Chen et al., 2022]

Short Story

Parametric Models

- BTL: $M_{ik} = \phi(a_i - b_k)$

Non-Parametric Models

- Isotonic: M_{π^*} isotonic
- Bi-Isotonic: $M_{\pi^* \eta^*}$ bi-isotonic

- No computational gap for **parametric models**

  e.g. [Chen et al., 2022]

- Mostly unknown for **non-parametric models**: computational gaps were conjectured

  e.g. [Flammarion et al., 2019, Mao et al., 2018]

Contributions

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Corollary for the bi-isotonic model ($M_{\pi^*\eta^*}$ bi-isotonic):

- Poly. time algo achieves better rates than state of the art [Mao et al., 2018, Liu and Moitra, 2020]

Existing Methods

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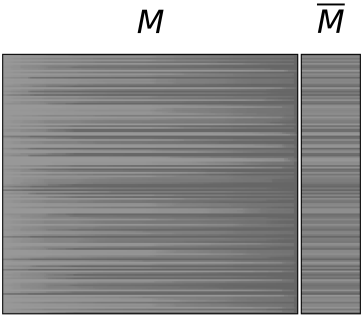
Global Average Comparison



Matrix M

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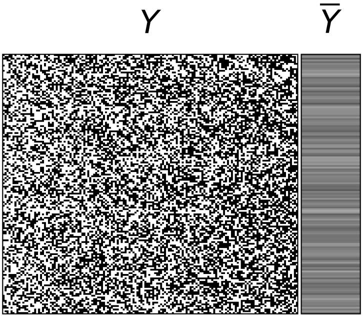
- Rank according to the averages:

$$\overline{Y}_i = \frac{1}{d}\sum_{k=1}^{d} Y_{ik}$$



Matrix Y (M in noise).

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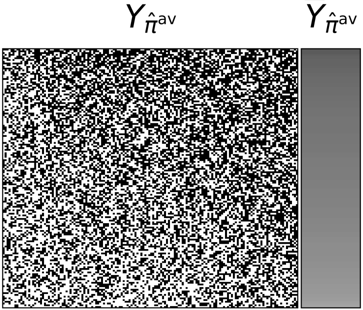

Matrix $Y_{\hat{\pi}^{\mathrm{av}}}$ (M in noise).

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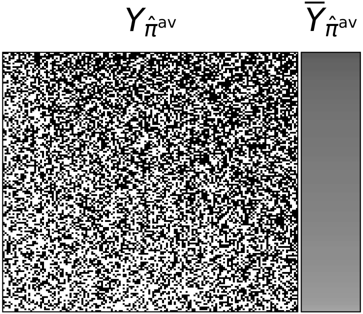

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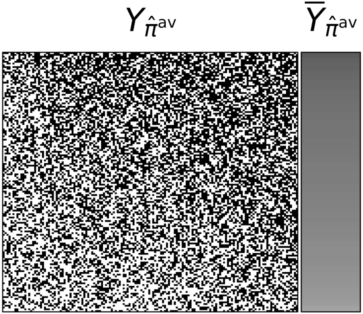
Guarantee on $\hat\pi^{\text{av}}$ [Shah et al., 2016]

$$\mathcal{R}^{\hat\pi^{\text{av}}}_{\text{perm}}(n,n) \in [\log^c(n)n^{3/2}, \log^C(n)n^{3/2}]$$



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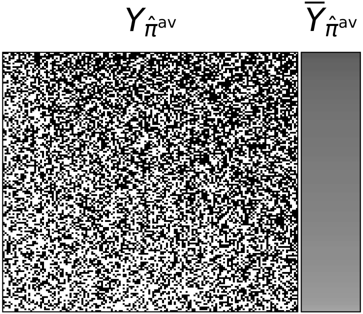

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- $\mathcal{R}^{\hat{\pi}^{\text{av}}}_{\text{perm}} \gtrsim n\sqrt{d}$

Existing Methods

- Simple Global Average Comparison
- Least-Square
- Poly. Time Methods only in the Bi-isotonic Model

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- $n^{4/3}$ still not optimal

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Polynomial Time Method in The Bi-Isotonic Model

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- Not optimal when we have partial observations ($\lambda \ll 1$)

Contributions

$$\mathcal{R}^{\hat{\pi}}_{\text{perm}} = \sup_{M,\pi^*} \mathbb{E}[\|M_{\hat{\pi}} - M_{\pi^*}\|_F^2]$$

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Theorem

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When $n = d$, we recover the $n^{7/6}$ rate in the easier bi-isotonic model

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- Perform a polylog number of iterations

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Permutation Estimation

Thresholded Graphs

$$\mathcal{G}(\gamma) = \{(i,j)\ :\ \mathcal{W}^{\mathrm{f}}_{ij} \geq \gamma\}$$

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- $\mathcal{G}(\gamma)$ is $\emptyset$ if γ is large
- $\mathcal{G}(0)$ is the complete graph

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$$\mathcal{G}(\gamma) = \{(i,j)\ :\ \mathcal{W}^{\mathrm{f}}_{ij} \geq \gamma\}$$

Minimal threshold

$$\hat{\gamma} = \inf_\gamma\{\mathcal{G}(\gamma) \text{ is acyclic}\}$$

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- $\mathcal{G}(0)$ is the complete graph

$$\mathcal{W} \in \mathbb{R}^{n \times n},\ \mathcal{W}_{ij} = -\mathcal{W}_{ji}$$

For all i, compute:

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4. $\mathcal{U}_{ij} = \langle Y_{i\cdot} - Y_{j\cdot}, u \rangle$

If $|\mathcal{U}_{ij}| \geq |\mathcal{W}_{ij}|$,
Update $\mathcal{W}_{ij} := \mathcal{U}_{ij}$

Final weighted graph $\mathcal{W}^{\mathrm{f}}$

Permutation Estimation

Thresholded Graphs

$$\mathcal{G}(\gamma) = \{(i,j)\ :\ \mathcal{W}^{\mathrm{f}}_{ij} \geq \gamma\}$$

Minimal threshold

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Estimator $\hat{\pi}$ is such that

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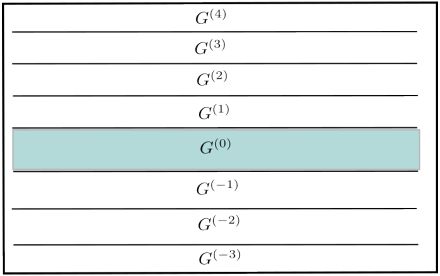

Choosing $\hat{Q}$

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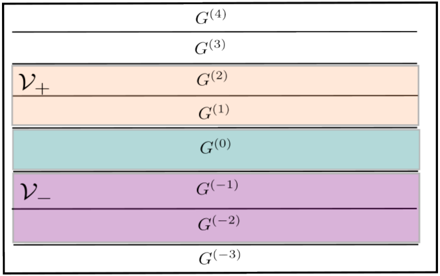

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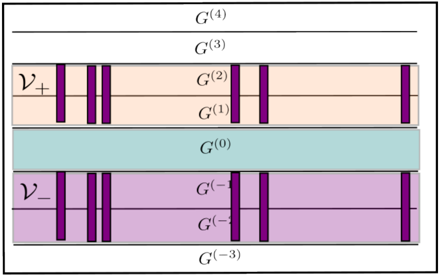

Set $\hat{Q}$

Reduce the dim. of columns to

$$\hat{Q} = \{k \ : \ \frac{1}{|\mathcal{V}^+|} \sum_{i \in \mathcal{V}^+} Y_{ik} - \frac{1}{|\mathcal{V}^-|} \sum_{i \in \mathcal{V}^-} Y_{ik} > h\}$$

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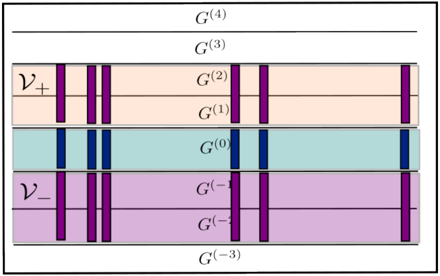

$Y(G^{(0)}, \hat{Q})$

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Spectral method on $Y(G^{(0)}, \hat{Q})$

If $|\mathcal{U}_{ij}| \geq |\mathcal{W}_{ij}|$,
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Conclusion

Isotonic Model

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- Results adapted to rectangular cases and partial observations
- Can be associated with isotonic regression for optimal estimation of the whole matrix M
- Improve existing poly. time rates in all the regimes in the easier bi-isotonic model

Crowdsourcing Problems with Unknown Labels

Vector of unknown labels $x^* \in \{-1, 1\}^d$

$$Y_{ik} = \begin{cases} x_k^*, & \text{with probability } M_{ik}, \\ -x_k^*, & \text{with probability } 1 - M_{ik}. \end{cases}$$

Objective: recover the vector of labels x^* and close the computational gap conjectured in [Shah et al., 2020]

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Objective: recover the vector of labels x^* and close the computational gap conjectured in [Shah et al., 2020]

Partial observations:

$$Y_{ik} = \pm x_k^* \text{ with probability } \lambda.$$

Spectral Method

Simple scenario in **isotonic** model:

$$M := M(G^{(0)}, \hat{Q}) = \frac{1}{2} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & h & 0 & h & 0 & 0 & h \\ 0 & h & h & 0 & h & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & h & 0 & h & 0 & 0 & h \end{pmatrix}$$

- Rank 1 matrix: spectral gap if h is large.
- Compute:

$$\hat{v} = \sup_{\|v\|=1} \|v^T Y\|_2^2 \ .$$

- In the isotonic model, use that

$$\|M - \overline{M}\|_{\text{op}} \geq \frac{1}{\log^C(nd)} \|M - \overline{M}\|_F \ .$$

- Iterate a spectral method a polylogarithmic number of time.

Parametric Models

Observation Model

$Y_{ik} = \mathrm{Bern}(M_{ik})$, with $M \in [0,1]^{n \times d}$

- Independent observations
- Poisson(1) observations per entry (i, k)

Noisy Sorting:
[Braverman and Mossel, 2008]

$$|M_{ik} - \frac{1}{2}| \geq \gamma \ ,$$

For some $\gamma > 0$

- Goal: Estimate π^* such that $M_{\pi^*(i+1), \pi^*(i)} \geq \frac{1}{2} + \gamma$

BTL: [Bradley and Terry, 1952]
Unknown vector $\theta \in \mathbb{R}^n$

$$M_{ik} = \frac{e^{(\theta_i - \theta_k)}}{1 + e^{(\theta_i - \theta_k)}}$$

- θ_i: ability of player i
- Goal: estimate a ranking π^* such that $\theta_{\pi^*(1)} \leq \cdots \leq \theta_{\pi^*(n)}$

Non-Parametric Models

Observation Model

$Y_{ik} = \text{Bern}(M_{ik})$, with $M \in [0,1]^{n \times d}$

- Independent observations
- Poisson(1) observations per entry (i, k)

SST: [Shah et al., 2016]

$\exists$ unknown permutation π^*:

$$M_{\pi^*,\pi^*} \text{ is bi-isotonic}$$

$$\text{e.g.: } \begin{pmatrix} 0.5 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.7 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

- $M_{ij} \geq 1/2$ implies $M_{ik} \geq M_{jk}$
- Goal: estimate π^*

Bi-Isotonic-2D: [Mao et al., 2018]

$\exists$ unknown permutations π^*, η^*:

$$M_{\pi^*,\eta^*} \text{ is bi-isotonic}$$

- η^* represents a ranking of the difficulty of the questions

Non-Parametric Bi-Isotonic-1D and Isotonic Models

Observation Model

$Y_{ik} = \mathrm{Bern}(M_{ik})$, with $M \in [0,1]^{n \times d}$

- Independent observations
- Poisson(1) observations per entry (i, k)

Bi-Isotonic-1D: [Mao et al., 2018]

$\exists$ unknown permutations π^*:

$$M_{\pi^*} \text{ is bi-isotonic}$$

- Corresponds to $\eta^* = \mathrm{id}$ in the bi-isotonic-2D model
- The questions are ordered by difficulty

Isotonic: [Flammarion et al., 2019]

$\exists$ unknown permutation π^*:

$$M_{\pi^*} \text{ is isotonic}$$

- No isotonicity constraint on the rows of M
- More flexible than the bi-isotonic 1D, 2D models and SST

Summary

Parametric Models

- BTL: $M_{ik} = \frac{e^{(\theta_i - \theta_k)}}{1+e^{(\theta_i - \theta_k)}}$
- Noisy Sorting: $|M_{ik} - \frac{1}{2}| \geq \gamma$

Non-Parametric Models

- SST: $M_{\pi^*\pi^*}$ is bi-isotonic
- Bi-Isotonic-2D: $M_{\pi^*\eta^*}$ is bi-isotonic
- Bi-Isotonic-1D: M_{π^*} is bi-isotonic
- Isotonic: M_{π^*} is isotonic

Sorting models by statistical difficulty:

$$\text{Bi-isotonic-1D} \prec \text{Bi-Isotonic-2D} \prec \text{Isotonic}$$

Rates in Isotonic and Bi-Isotonic-1D Models

Isotonic Model:

	$n \lesssim d^{3/2}$	$d^{3/2} \lesssim n$
$\mathcal{R}^*_{\text{perm}}$	$n^{2/3}\sqrt{d}$	n
$\mathcal{R}^*_{\text{est}}$	$n^{1/3}d$	n

Bi-Isotonic-1D Model:

	$n \lesssim d^{1/3}$	$d^{1/3} \lesssim n \lesssim d$	$d \lesssim n$
$\mathcal{R}^*_{\text{perm}}$	$nd^{1/6}$	$n^{3/4}d^{1/4}$	n
$\mathcal{R}^*_{\text{est}}$	$nd^{1/3}$	$\sqrt{nd}$	n

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